International Journal of Theoretical Physics, Vol. 18, No. 11, 1979

Quantum Electrodynamics of One Scalar Particle

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Received September 17, 1979

The theory of the interaction between a complex scalar field and the electromagnetic field is presented with initial and final conditions that allow an interpretation in the context of the relativistic quantum mechanics of a single charged scalar particle. Included are particle scattering, antiparticle scattering, pair creation, and pair annihilation due to a classical dynamical electromagnetic field. The equations of motion are solved by a perturbation expansion, which does not lead to the troublesome divergent terms of quantum field theory.

1. INTRODUCTION

It is a well-known result in nonrelativistic quantum mechanics that the theory of a (second) quantized Schrödinger field is equivalent to the quantum mechanics of many particles. A similar claim is made by Bjorken and Drell (1964, 1965) for the relativistic theory, but since there are many versions of each type of formulation which are often not mathematically rigorous, the question of equivalence is not even well defined.

We have emphasized (Marx, 1969, 1970a, 1970b) the probabilistic interpretation of wave functions in relativistic quantum mechanics and causal boundary conditions at finite initial and final times, t_i and t_f. Equations are relativistically covariant, but the separation of a wave function into positive- and negative-frequency parts and the specification of boundary conditions introduces an observer (Marx, 1970c), usually but not necessarily tied to his rest frame. A relativistic wave function for one "particle"[1] thus has a part that propagates forward in time and is specified

[1]We use "particle" to refer collectively to a particle and its antiparticle, which in our formulation are a single entity in different propagation states.

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at t_i, the probability amplitude for the particle, and a part that propagates backward in time and is specified at t_f, the probability amplitude for the antiparticle. An electromagnetic interaction changes these amplitudes, giving rise to pair creation and annihilation in addition to particle and antiparticle scattering. Thus, a wave function of a single 4-vector variable represents all four of these processes, and it is *not* necessary to go to quantum field theory when the number of particles and the number of antiparticles are not independently conserved. As long as the charge is conserved, all such processes can be represented by amplitudes with a fixed number of "particles" in the form of Dirac's many-times formalism.

A second quantization of this theory is possible, as shown for the spinor field in Marx (1972b), but it also requires one time variable per "particle." Also, the quantum field theory and the relativistic quantum mechanics versions of our formulation are very close, as is the case in the nonrelativistic theory. We can define a time displacement matrix and a scattering matrix (Marx, 1972b), but in this paper we deal with the probability amplitudes that provide a complete description of the system.

The relativistic quantum mechanics of scalar particles in an external electromagnetic field is obtained in a straightforward way from the Klein–Gordon equation. For a bispinor field, the Dirac equation leads to a positive or negative definite conserved density that cannot be interpreted as a charge density, and the equation has to be modified along the lines suggested by quantum field theory.

Not only are charged particles affected by given electromagnetic fields, but they interact among themselves and also radiate. The nonrelativistic Coulomb interaction is difficult to generalize to a relativistic theory, especially considering that the classical theory of the electromagnetic interactions of charged particles is still an open subject lacking a Hamiltonian approach. We introduce a dynamical electromagnetic field by using the Klein–Gordon equation for the scalar field and adding Maxwell's equations with the appropriate sources. The simplest case involves a single "particle," and this is the problem that we address here. We thus describe Compton scattering by a particle and by an antiparticle, pair creation and pair annihilation.

In Section 2, we find the equations of motion and discuss gauge conditions and boundary conditions. We choose to work in the Coulomb gauge, and we develop a perturbation expansion of the solution in the case that includes particle scattering and pair annihilation in Section 3. Then, in Section 4, we discuss time reflection, which includes charge conjugation, and we conclude with some remarks in Section 5.

We use the time-favoring metric in space–time and the modified summation convention for repeated Greek subindices that range from 0 to

3. We use natural units such that $\hbar$, c, ε_0, and μ_0 are all equal to 1, and the notation is further explained in the references.

2. INTERACTING SCALAR AND ELECTROMAGNETIC FIELDS

The wave function for a charged spinless particle is a complex scalar field, ϕ. The electromagnetic field, $F_{\mu\nu}$, can be expressed in terms of the potentials, A_μ, by

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \tag{2.1}$$

The equations of motion can be derived from the Lagrangian density

$$\mathfrak{L} = (D_\mu^* \phi^*) D_\mu \phi + m^2 \phi^* \phi - \tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} \tag{2.2}$$

where

$$D_\mu = \partial_\mu + ieA_\mu \tag{2.3}$$

We obtain the Klein–Gordon equation

$$(D^2 + m^2)\phi = 0 \tag{2.4}$$

and the inhomogeneous Maxwell equations

$$F_{\mu\nu,\nu} = j_\mu \tag{2.5}$$

where

$$j_\mu = ie\left[\phi^* D_\mu \phi - (D_\mu^* \phi^*)\phi\right] \tag{2.6}$$

The homogeneous Maxwell equations are satisfied as a consequence of equations (2.1).

The specification of boundary conditions for the scalar field is closely related to its physical interpretation and the choice of a Green's function to solve the inhomogeneous Klein–Gordon equation

$$(\partial^2 + m^2)\phi(x) = \omega(x) \tag{2.7}$$

We choose the causal Green's function or Feynman propagator and use

Green's theorem to write the solution in the form

$$\phi(x) = -\int d^3x' \int_{t_i}^{t_f} dt' \Delta_F(x'-x)\omega(x')$$

$$-\int d^3x' \Delta^{(+)}(x-x') \overleftrightarrow{\partial}_0' \phi(x')|_{t'=t_i}$$

$$-\int d^3x' \Delta^{(-)}(x-x') \overleftrightarrow{\partial}_0' \phi(x')|_{t'=t_f} \tag{2.8}$$

where t is an intermediate time,

$$\Delta_F(x) = \theta(t)\Delta^{(+)}(x) - \theta(-t)\Delta^{(-)}(x) \tag{2.9}$$

$$\Delta^{(\pm)}(x) = \mp i(2\pi)^{-3} \int d^3k (2k_0)^{-1} \exp[\mp ik\cdot x] \tag{2.10}$$

$$k_0 = +(\mathbf{k}^2 + m^2)^{1/2} \tag{2.11}$$

and $\theta(t)$ is the unit step function. The Green's function, Δ_F, satisfies

$$(\partial^2 + m^2)\Delta_F(x) = -\delta(x) \tag{2.12}$$

while $\Delta^{(\pm)}$ are solutions of the homogeneous Klein–Gordon equation which also satisfy

$$\partial_0 \Delta^{(\pm)}(x) = \mp i\tilde{E}\Delta^{(\pm)}(x) \tag{2.13}$$

$$2\tilde{E}\Delta^{(\pm)}(\mathbf{x}, 0) = \mp i\delta(\mathbf{x}) \tag{2.14}$$

where $\tilde{E}$ is the integral operator

$$\tilde{E} = (-\nabla^2 + m^2)^{1/2} \tag{2.15}$$

We use equation (2.13) and integration by parts to rewrite equation (2.8) as

$$\phi(x) = -\int d^3x' \int_{t_i}^{t_f} dt' \Delta_F(x'-x)\omega(x')$$

$$+ i\int d^3x' \left[\tilde{E}'\Delta^{(+)}(\mathbf{x}-\mathbf{x}', t-t_i)\right]\left[\phi(\mathbf{x}', t_i) + i(\tilde{E}')^{-1}\dot{\phi}(\mathbf{x}', t_i)\right]$$

$$- i\int d^3x' \left[\tilde{E}'\Delta^{(-)}(\mathbf{x}-\mathbf{x}', t-t_f)\right]\left[\phi(\mathbf{x}', t_f) - i(\tilde{E}')^{-1}\dot{\phi}(\mathbf{x}', t_f)\right] \tag{2.16}$$

The last two terms show what combinations of the function and its time derivative have to be specified at the initial and final times. If we assume that the sources vanish at these times, the function ϕ can be decomposed into its positive- and negative-frequency parts,

$$\phi^{(\pm)}(x)=\tfrac{1}{2}\left[\phi(x)\pm i\tilde{E}^{-1}\dot{\phi}(x)\right] \tag{2.17}$$

which are related to the probability amplitudes $g^{(\pm)}(x)$ by

$$g^{(\pm)}(x)=(2\tilde{E})^{1/2}\phi^{(\pm)}(x) \tag{2.18}$$

When the electromagnetic field does not vanish, the probability amplitudes are generalized to

$$g^{(\pm)}(x)=(\tilde{E}/2)^{1/2}(1\pm i\tilde{E}^{-1}D_0)\phi(x) \tag{2.19}$$

so that the conserved charge is always

$$Q=e\int d^3x\left[|g^{(+)}(x)|^2-|g^{(-)}(x)|^2\right] \tag{2.20}$$

Although we can specify two arbitrary functions $g^{(+)}(\mathbf{x},t_i)$ and $g^{(-)}(\mathbf{x},t_f)$, considerations of the physical interpretation limit us to two cases. Either we specify the particle amplitude at the initial time normalized so that

$$\int d^3x|g^{(+)}(\mathbf{x},t_i)|^2=1 \tag{2.21}$$

and assume that the antiparticle amplitude vanishes at the final time, or we give the latter so that

$$\int d^3x|g^{(-)}(\mathbf{x},t_f)|^2=1 \tag{2.22}$$

and assume that there is no particle at the initial time. The purpose of the dynamical calculations is to find both $g^{(+)}(\mathbf{x},t_f)$ and $g^{(-)}(\mathbf{x},t_i)$. More generally, we define $\phi^{(\pm)}(x)$ in the case of the inhomogeneous Klein–Gordon equation to be

$$\begin{aligned}\phi^{(+)}(x)={}&-\int d^3x'\int_{t_i}^{t}dt'\Delta^{(+)}(x-x')\omega(x')\\&+i\int d^3x'\left[2\tilde{E}'\Delta^{(+)}(\mathbf{x}-\mathbf{x}',t-t_i)\right]\phi^{(+)}(\mathbf{x}',t_i)\end{aligned} \tag{2.23}$$

$$\begin{aligned}\phi^{(-)}(x)={}&\int d^3x'\int_{t}^{t_f}dt'\Delta^{(-)}(x-x')\omega(x')\\&-i\int d^3x'\left[2\tilde{E}'\Delta^{(-)}(\mathbf{x}-\mathbf{x}',t-t_f)\right]\phi^{(-)}(\mathbf{x}',t_f)\end{aligned} \tag{2.24}$$

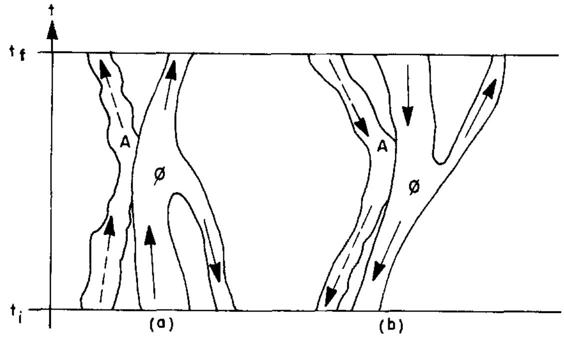

Fig. 1. Representation of the interaction between a charged spinless particle and electromagnetic radiation when (a) a particle is specified at the initial time or (b) an antiparticle is specified at the final time.

The normalization of the given probability amplitude and charge conservation imply that

$$\int d^3x |g^{(+)}(\mathbf{x},t_f)|^2 + \int d^3x |g^{(-)}(\mathbf{x},t_i)|^2 = 1 \tag{2.25}$$

In our physical interpretation, measurements are carried out on particles at the final time and on antiparticles at the initial time. If a particle is specified at the initial time, the first term is the probability that this particle would be scattered and the second term is the probability that the particle would be annihilated. If an antiparticle is specified at the final time, these terms correspond to pair creation and antiparticle scattering, respectively. These processes are shown in Figure 1.

Equations (2.5) for the electromagnetic potentials comprise three equations of motion and one constraint. The fields are invariant under gauge transformations of the second kind,

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \Lambda_{,\mu}(x) \tag{2.26}$$

where Λ is an essentially arbitrary real function. We also have to change the phase of the scalar field by

$$\phi(x) \rightarrow \phi'(x) = \phi(x)\exp[-ie\Lambda(x)] \tag{2.27}$$

to preserve the form of the Klein–Gordon equation. We have shown (Marx, 1970c) that there is a gauge-dependent part of the potential which vanishes in the Coulomb or radiation gauge and a physically meaningful part that is invariant.

We first consider a Lorentz gauge, which preserves manifest relativistic covariance, and restrict the potentials by

$$\partial_\mu A_\mu = 0 \tag{2.28}$$

Equations (2.2) and (2.5) reduce to

$$(\partial^2 + m^2)\phi = -2ieA_\mu \phi_{,\mu} - e^2 A^2 \phi \tag{2.29}$$

$$\partial^2 A_\mu = j_\mu \tag{2.30}$$

In the case of retarded boundary conditions, we have to specify A_μ and $\dot{A}_\mu$ at the initial time. These eight functions are constrained by the Lorentz condition (2.28) and Maxwell's equation

$$\nabla \cdot \mathbf{E} = -\Delta \cdot \dot{\mathbf{A}} - \nabla^2 A_0 = j_0 \tag{2.31}$$

The charge density j_0 depends not only on A_0 and $g^{(+)}$, which are known, but also on $g^{(-)}$, which has to be determined by solving the equations of motion. We thus choose to work in the Coulomb gauge and have the vector potential satisfy

$$\nabla \cdot \mathbf{A} = 0 \tag{2.32}$$

equations (2.2) and (2.5) reduce to

$$(\partial^2 + m^2)\phi = -2ieA_\mu \phi_{,\mu} + i\dot{A}_0 \phi - e^2 A^2 \phi \tag{2.33}$$

$$\nabla^2 A_0 = -j_0 \tag{2.34}$$

$$\partial^2 \mathbf{A} = \mathbf{j}_t \tag{2.35}$$

where $\mathbf{j}_t$ is the transverse part of the current density $\mathbf{j}$, given by

$$\mathbf{j}_t = \frac{1}{4\pi} \nabla \wedge \left[\nabla \wedge \int \frac{\mathbf{j}(\mathbf{x}', t) d^3x'}{|\mathbf{x} - \mathbf{x}'|} \right] \tag{2.36}$$

an unfortunate consequence of (2.32).

The causal Green's function for the electromagnetic potentials appears naturally in quantum field theory, and it is sometimes taken over into relativistic quantum mechanics (Bjorken and Drell, 1964). We do not use it here because the Feynman propagator, $D_F(x)$, is complex and thus not appropriate for the real fields A_μ. Furthermore, the positive- or the negative-frequency part of a real field requires the specification of both the field and its time derivative, which cannot be given at two different times. Consequently, we select the retarded Green's function, which requires the values of the function and its time derivative at the initial time, in the case of a particle given at this time. Invariance under time reflection then would lead to the field specification at the final time when an antiparticle is given and the use of the advanced Green's function. The retarded Green's function,

$$D_R(x) = -(4\pi|\mathbf{x}|)^{-1}\delta(t-|\mathbf{x}|)\theta(t) \tag{2.37}$$

is real, and it satisfies

$$\partial^2 D_R(x) = -\delta(x) \tag{2.38}$$

Green's theorem then gives the field at later times,

$$\mathbf{A}(x) = -\int d^3x' \int_{t_i}^{t} dt' D_R(x-x')\mathbf{j}_t(x') - \int d^3x' D_R(\mathbf{x}-\mathbf{x}', t-t_i)\overleftrightarrow{\partial}_0' \mathbf{A}(\mathbf{x}', t_i) \tag{2.39}$$

If we substitute the expression (2.36) for $\mathbf{j}_t$ in the first term and integrate by parts,[2] we see that we do not have to compute the transverse part of $\mathbf{j}$ if we perform a similar operation with the Green's function. We write

$$\mathbf{A}(x) = -\int d^3x' \int_{t_i}^{t} dt' \mathsf{D}_R(x-x')\cdot\mathbf{j}(x') - \int d^3x' \left[D_R(\mathbf{x}-\mathbf{x}', t-t_i)\dot{\mathbf{A}}(\mathbf{x}', t_i) + \dot{D}_R(\mathbf{x}-\mathbf{x}', t-t_i)\mathbf{A}(\mathbf{x}', t_i) \right] \tag{2.40}$$

where the components of the tensor Green's function, D_R, are

$$D_{Rij}(x-x') = (4\pi)^{-1}\left(\partial_i\partial_j - \nabla^2\delta_{ij}\right)\int d^3x'' |\mathbf{x}'-\mathbf{x}''|^{-1} D_R(\mathbf{x}-\mathbf{x}'', t-t') \tag{2.41}$$

The scalar potential, A_0, obeys the Laplace equation, and we can write the

[2]We are really dealing with distributions, and we are assuming that continuity and differentiability conditions on $\mathbf{j}(x)$ are satisfied.

solution

$$A_0(x)=(4\pi)^{-1}\int d^3x'|\mathbf{x}-\mathbf{x}'|^{-1}j_0(\mathbf{x}',t) \tag{2.42}$$

Thus, to study the interaction between an incoming particle and a radiation field, we specify

$$\phi^{(+)}(\mathbf{x},t_i)=f(\mathbf{x}), \qquad \phi^{(-)}(\mathbf{x},t_f)=0 \tag{2.43}$$

$$\mathbf{A}(\mathbf{x},t_i)=\mathbf{a}(x), \qquad \dot{\mathbf{A}}(\mathbf{x},t_i)=\mathbf{b}(x) \tag{2.44}$$

and the complex function f and real vector functions $\mathbf{a}$ and $\mathbf{b}$ have to satisfy

$$\int d^3x|(2\tilde{E})^{-1/2}f(\mathbf{x})|^2=1 \tag{2.45}$$

$$\nabla\cdot\mathbf{a}(\mathbf{x})=0, \qquad \nabla\cdot\mathbf{b}(\mathbf{x})=0 \tag{2.46}$$

We then solve the Klein–Gordon and Maxwell equations to find $\phi^{(+)}(\mathbf{x},t_f)$, $\phi^{(-)}(\mathbf{x},t_i)$, $\mathbf{A}(\mathbf{x},t_f)$, and $\dot{\mathbf{A}}(\mathbf{x},t_f)$, although we can obtain ϕ and A_μ at all intermediate times as well.

3. PERTURBATION EXPANSION

The problem stated at the end of Section 2 can be solved approximately if we expand the fields in powers of the electric charge, that is,

$$\phi(x)=\sum_{i=0}^{\infty} e^i\phi^{(i)}(x) \tag{3.1}$$

$$A_\mu(x)=\sum_{i=0}^{\infty} e^iA_\mu^{(i)}(x) \tag{3.2}$$

We are assuming that these expansions are possible, although we know that there could be branch points that would lead to divergences in some order.

We solve the equations of motion (2.33)–(2.35) in each order of e. They have the form

$$(\partial^2+m^2)\phi^{(i)}(x)=\omega^{(i)}(x) \tag{3.3}$$

$$\nabla^2A_0^{(i)}(x)=-j_0^{(i)}(x) \tag{3.4}$$

$$\partial^2\mathbf{A}^{(i)}(x)=\mathbf{j}_t^{(i)}(x) \tag{3.5}$$

We also have to make sure that the constraint is satisfied by the initial conditions in each order to remain in the Coulomb gauge.

The sources all vanish in zeroth order, so the fields obey homogeneous equations and they have to satisfy the given boundary conditions. Equations (2.46) ensure that the gauge condition is satisfied. The zeroth-order fields are

$$\phi^{(0)}(x)=i\int d^3x'\left[2\tilde{E}'\Delta^{(+)}(\mathbf{x}-\mathbf{x}',t-t_i)\right]f(\mathbf{x}') \tag{3.6}$$

$$A_0^{(0)}(x)=0 \tag{3.7}$$

$$\mathbf{A}^{(0)}(x)=-\int d^3x'\left[D_R(\mathbf{x}-\mathbf{x}',t-t_i)\mathbf{b}(\mathbf{x}')+\dot{D}_R(\mathbf{x}-\mathbf{x}',t-t_i)\mathbf{a}(\mathbf{x}')\right] \tag{3.8}$$

The higher-order contributions to the fields satisfy homogeneous boundary conditions, and the sources are functions of the lower-order fields. Up to the fourth order, the sources are

$$\omega^{(1)}=-2i\mathbf{A}^{(0)}\cdot\nabla\phi^{(0)} \tag{3.9}$$

$$j_\mu^{(1)}=i\left(\phi^{(0)*}\phi_{,\mu}^{(0)}-\phi_{,\mu}^{(0)*}\phi^{(0)}\right) \tag{3.10}$$

$$\omega^{(2)}=-2i\left(A_\mu^{(1)}\phi_{,\mu}^{(0)}+A_\mu^{(0)}\phi_{,\mu}^{(1)}\right)+i\dot{A}_0^{(1)}\phi^{(0)}+\mathbf{A}^{(0)2}\phi^{(0)} \tag{3.11}$$

$$j_\mu^{(2)}=i\left(\phi^{(1)*}\phi_{,\mu}^{(0)}+\phi^{(0)*}\phi_{,\mu}^{(1)}-\text{c.c.}\right)-2\phi^{(0)*}\phi^{(0)}A_\mu^{(0)} \tag{3.12}$$

$$\begin{aligned}\omega^{(3)}=&-2i\left(A_\mu^{(2)}\phi_{,\mu}^{(0)}+A_\mu^{(1)}\phi_{,\mu}^{(1)}+A_\mu^{(0)}\phi_{,\mu}^{(2)}\right)\\&+i\left(\dot{A}_0^{(2)}\phi^{(0)}+\dot{A}_0^{(1)}\phi^{(1)}\right)+\mathbf{A}^{(0)2}\phi^{(1)}+2\mathbf{A}^{(0)}\cdot\mathbf{A}^{(1)}\phi^{(0)}\end{aligned} \tag{3.13}$$

$$\begin{aligned}j_\mu^{(3)}=&\,i\left(\phi^{(2)*}\phi_{,\mu}^{(0)}+\phi^{(1)*}\phi_{,\mu}^{(1)}+\phi^{(0)*}\phi_{,\mu}^{(2)}-\text{c.c.}\right)\\&-2(\phi^{(1)*}\phi^{(0)}+\phi^{(0)*}\phi^{(1)})A_\mu^{(0)}-2\phi^{(0)*}\phi^{(0)}A_\mu^{(1)}\end{aligned} \tag{3.14}$$

$$\begin{aligned}\omega^{(4)}=&-2i\left(A_\mu^{(3)}\phi_{,\mu}^{(0)}+A_\mu^{(2)}\phi_{,\mu}^{(1)}+A_\mu^{(1)}\phi_{,\mu}^{(2)}+A_\mu^{(0)}\phi_{,\mu}^{(3)}\right)\\&+i\left(\dot{A}_0^{(3)}\phi^{(0)}+\dot{A}_0^{(2)}\phi^{(1)}+\dot{A}_0^{(1)}\phi^{(2)}\right)+\mathbf{A}^{(0)2}\phi^{(2)}\\&+2\mathbf{A}^{(0)}\cdot\mathbf{A}^{(1)}\phi^{(1)}-(A^{(1)2}-2\mathbf{A}^{(0)}\cdot\mathbf{A}^{(2)})\phi^{(0)}\end{aligned} \tag{3.15}$$

$$\begin{aligned}j_\mu^{(4)}=&\,i\left(\phi^{(3)*}\phi_{,\mu}^{(0)}+\phi^{(2)*}\phi_{,\mu}^{(1)}+\phi^{(1)*}\phi_{,\mu}^{(2)}+\phi^{(0)*}\phi_{,\mu}^{(3)}-\text{c.c.}\right)\\&-2(\phi^{(2)*}\phi^{(0)}+\phi^{(1)*}\phi^{(1)}+\phi^{(0)*}\phi^{(2)})A_\mu^{(0)}\\&-2(\phi^{(1)*}\phi^{(0)}+\phi^{(0)*}\phi^{(1)})A_\mu^{(1)}-2\phi^{(0)*}\phi^{(0)}A_\mu^{(2)}\end{aligned} \tag{3.16}$$

These source terms are then substituted in equations (2.16), (2.40), and (2.42) to obtain the fields to the desired order in perturbation theory. Since the boundary conditions are already satisfied by the terms of order zero, only the integral over the sources is present in higher orders. These integrals can be represented by Feynman diagrams, and they differ from those obtained in quantum field theory. Here they represent dynamical processes in space–time, and the finite elapsed time precludes four-dimensional Fourier transforms. In Figures 2–4, we show some of the diagrams that contribute to the scattered particle field, the scattered electromagnetic field, and the antiparticle field at the initial time that represents pair annihilation. The terms pictured in Figure 2 come from the sources given in equations (3.9) and (3.10), those in Figure 3 come from equations (3.11) and (3.12), and the other expressions give similar diagrams for higher-order contributions. The electromagnetic interaction represents both the constant-time scalar potential and the vector potential that propagates with the speed of light. The intermediate scalar field corresponds to both particle and antiparticle propagation, that is, the end points can have any time ordering. There are a number of differences between these diagrams and the ones that come from the usual theory of quantized fields. Although we are dealing with a single scalar field of one four-vector variable, the particle lines can appear repeatedly in the diagrams, as shown in Figure 4e. Similarly, the incoming and intermediate electromagnetic fields can appear several times, as in Figures 4a and 4e. All "particle" lines originate at the initial time and terminate at either the initial or the final time. The electromagnetic field emitted by the transverse part of the current propagates only forward in time, and the source is always the product of a field and the complex conjugate of a field. On the other hand, the absorption of the electromagnetic field appears only in terms where the field, and not its complex conjugate, is a source for a higher-order field. These observations

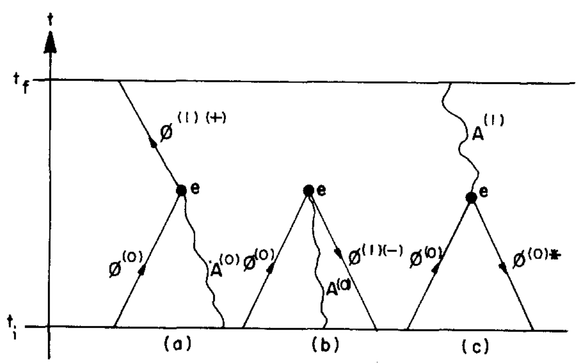

Fig. 2. Feynman diagrams for the first-order contributions to (a) particle scattering, (b) pair annihilation, and (c) emission of radiation.

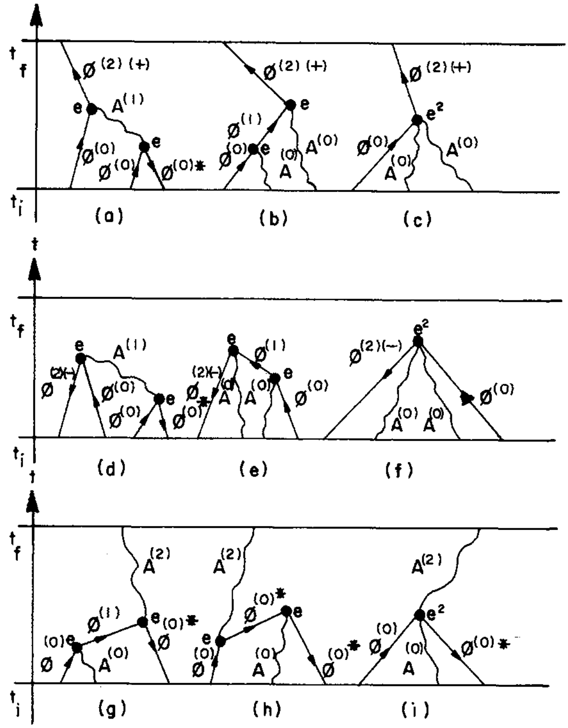

Fig. 3. Second-order contributions to (a)–(c) particle scattering, (d)–(f) pair annihilation, and (g)–(i) emission of radiation.

show why troublesome diagrams such a those in Figure 5 do not appear in this theory, which thus is free from these basic divergent terms.

4. TIME REFLECTION

The theory of the complex scalar field interacting with the electromagnetic field is invariant under time reflection, which is the improper antichronous Lorentz transformation given by

$$(a_\mu{}^\nu)=\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{4.1}$$

As a consequence, t is replaced by $-t$, the scalar potential also changes

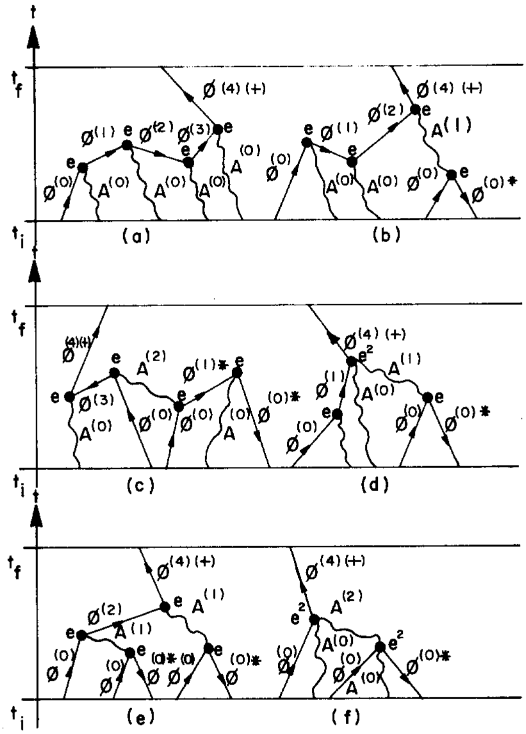

Fig. 4. Some of the fourth-order contributions to particle scattering.

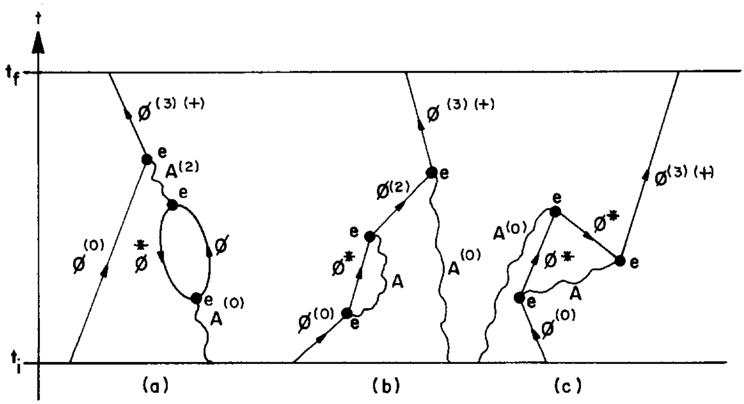

Fig. 5. Third-order diagrams that do *not* appear in this theory and that correspond to (a) vacuum polarization, (b) mass renormalization, and (c) charge renormalization or vertex correction.

sign, and the scalar field is not affected but for the change in the argument. As a consequence, particle and antiparticle amplitudes are interchanged, as well as the initial and final times. This transformation is also called Schwinger time reversal as opposed to Wigner time reversal or reversal of the direction of motion, in which the field is replaced by its complex conjugate to change back the roles of particles and antiparticles. We note that the Lagrangian density (2.2) is invariant under time reflection without a change in the sign of e.

The specification of boundary conditions is affected by time reflection. We have

$$\Delta^{(\pm)}(\mathbf{x},-t)=-\Delta^{(\mp)}(\mathbf{x},t) \tag{4.2}$$

$$\Delta_F(\mathbf{x},-t)=\Delta_F(\mathbf{x},t) \tag{4.3}$$

$$D_R(\mathbf{x},-t)=D_A(\mathbf{x},t) \tag{4.4}$$

which show that the scalar field behaves in the same manner as before, with the positive- and negative-frequency parts interchanged, while the retarded electromagnetic field becomes an advanced field and its boundary values are given at the final time. The problem solved in Section 3 is changed as follows. An antiparticle amplitude is given at the final time, together with the electromagnetic vector potential and its time derivative, and the particle amplitude at the initial time is set equal to zero. The causal Green's function can then be used to find a perturbation expansion of the antiparticle amplitude at the initial time, which represents the antiparticle scattering, and the particle amplitude at the final time, which corresponds to pair creation. Concurrently, the vector potential is determined at the initial time by means of the advanced Green's function, while the scalar potential can be found as a solution of Laplace's equation. These processes are represented in Figure 1b, and the diagrams in Figures 2–4 have to be turned upside down. We have found a similar association of retarded and advanced fields with particles and antiparticles, respectively, in a classical theory of electromagnetism (Marx, 1976a).

5. CONCLUDING REMARKS

We have presented the equations of motion and discussed the boundary conditions for interacting complex scalar and electromagnetic fields. We have also shown how to find a solution using perturbation expansions. The physical interpretation of this problem in terms of probability amplitudes for charged spinless particles and antiparticles and in terms of electromagnetic radiation forms the basic framework of relativistic quantum mechanics.

The given "particle" and the given radiation can be directed at each other either from the past or from the future. This point of view (Walter and Marx, 1971) is analogous to the one-dimensional potential barrier in nonrelativistic quantum mechanics, where particles come either from the right or from the left. When the particle and the radiation come from the past, the former can be "transmitted" or scattered, or it can be "reflected" or turned around in time, a process that an observer would interpret as pair annihilation; in either case, radiation is present at the final time. In the time-reflected process, an antiparticle and radiation come in from the future, and the antiparticle can be scattered or turned around in time, that is, pair creation can occur; radiation then is found at the initial time. There is no intrinsic difference between electromagnetic fields obtained by means of retarded or advanced Green's functions, but the form of the solutions of Maxwell's equations depends on the specification of boundary conditions.

There are several obvious generalizations of this problem, but they are beset with serious difficulties. Of more interest than spinless particles are electrons, that is, spin-$\frac{1}{2}$ particles that are usually represented by a bispinor field that obeys the Dirac equation. We cannot interpret this field in terms of probability amplitudes for electrons and positrons because the conserved current density has a charge-density component that does not change sign. We have tried to overcome this problem in a number of ways (Marx, 1970b, d, 1972b, 1974a, c, 1976b), with only limited success. The relativistic quantum mechanics of several particles involves one time variable for each particle (Marx, 1970a, 1972b), which does not lend itself to a Hamiltonian formalism. The classical theory of the electromagnetic interaction of charged point particles is still evolving, is not free of divergent terms, and does not have a canonical formulation; it is hard to see how we can use this theory as a starting point for the interaction between two or more particles in relativistic quantum mechanics. To generate an electromagnetic interaction, not between the particles, but between charge densities formed from the probability amplitudes (Marx, 1972a), we find problems with many-particle amplitudes. Alternatively, the electromagnetic field would have to be generalized to accept sources of the form (Marx, 1970a)

$$\begin{aligned} j_{\mu\nu\lambda}(x_1,x_2,x_3) = i(&\phi^*_{,\mu\nu\lambda}\phi - \phi^*_{,\mu\nu}\phi_{,\lambda} - \phi^*_{,\nu\lambda}\phi_{,\mu} \\ &- \phi^*_{,\lambda\mu}\phi_{,\nu} + \phi^*_{,\mu}\phi_{,\nu\lambda} + \phi^*_{,\nu}\phi_{\lambda\mu} + \phi^*_{,\lambda}\phi_{,\mu\nu} - \phi^*\phi_{,\mu\nu\lambda}) \end{aligned} \tag{5.1}$$

in the case of a three-particle wave function $\phi(x_1,x_2,x_3)$. Even a many-times formalism for the Coulomb interaction in the nonrelativistic approximation (Marx, 1974b) is not free of problems. The quantization of the electromagnetic field is made difficult by the questions of gauge invariance and constraints. If only the radiation field is quantized in a gauge-independent manner (Goldberg and Marx, 1968), the procedure is observer dependent, and we still have to deal with the Coulomb interaction. A more covariant formulation has to rely on special procedures such as those suggested by Gupta and Bleuler,[3] which single out the component A_0 of the electromagnetic potential and involve reference to unphysical states.

It is also not straightforward to compare our results to those obtained in terms of scattering amplitudes. We have a dynamical formalism where the time variable has a special role and is limited to a finite interval. We do not see any advantages in introducing divergences and ill-defined quantities by going to infinite times and plane-wave states in momentum space, although wave packets given by $a^{(\pm)}(\mathbf{k},t)$ have been used in several of our papers.

What we have here is a perturbation expansion that is free of the usual divergent terms of quantum field theory. The mass and the charge are the physical parameters, and no renormalization procedure is required. This is one more step in our program to reformulate quantum electrodynamics without divergent quantities.

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[3]This formalism and references to the original papers can be found in most standard texts on the quantum theory of fields. Throughout this paper, we have refrained from giving references to well-known formulations found in many books and articles.